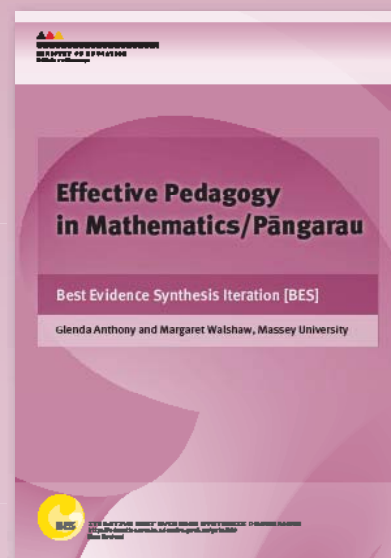# Expect students to be accountable for thinking through the mathematics involved in a problem

This is one of a series of cases that illustrate the findings of the best evidence syntheses (BESs). Each is designed to support the professional learning of educators, leaders and policy makers.

Effective Pedagogy in Mathematics/Pāngarau

Best Evidence Synthesis Iteration [BES]

Glenda Anthony and Margaret Walshaw, Massey University

BES cases: Insight into what works

The best evidence syntheses (BESs) bring together research evidence about 'what works' for diverse (all) learners in education. Recent BESs each include a number of cases that describe actual examples of professional practice and then analyse the findings. These cases support educators to grasp the big ideas behind effective practice at the same time as they provide vivid insight into their application.

Building as they do on the work of researchers and educators, the cases are trustworthy resources for professional learning.

Using the BES cases

The BES cases overview provides a brief introduction to each of the cases. It is designed to help you quickly decide which case or cases could be helpful in terms of your particular improvement priorities.

Use the cases with colleagues as catalysts for reflecting on your own professional practice and as starting points for delving into other sources of information, including related sections of the BESs. To request copies of the source studies, use the Research Behind the BES link on the BES website.

The conditions for effective professional learning are described in the Teacher Professional Learning and development BES and condensed into the ten principles found in the associated International Academy of Education summary (Timperley, 2008).

Note that, for the purpose of this series, the cases have been re-titled to more accurately signal their potential usefulness.

Responsiveness to diverse (all) learners

Use the BES cases and the appropriate curriculum documents to design a response that will improve student outcomes
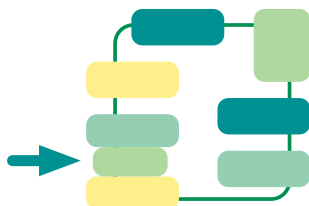
The different BESs consistently find that any educational improvement initiative needs to be responsive to the diverse learners in the specific context. Use the inquiry and knowledge-building cycle tool to design a collaborative approach to improvement that is genuinely responsive to your learners

Expect students to be accountable for thinking through the mathematics involved in a problem

The mathematics BES emphasises the importance of basing pedagogical practices on socio-mathematical norms that collectively establish expectations about what mathematical thinking is and is not.

Through the use of contrasting examples from four different teachers, this case illustrates important differences between effective and less effective teaching. Above all, effective teaching emphasises conceptual thinking, not superficial sharing of ideas and strategies, and mathematical argumentation, not procedural description/summary.

While this case is located in grades 4–5, it is relevant to all teachers of mathematics, including secondary school teachers.

See also BES Exemplar 1: *Developing communities of mathematical inquiry.*

Respectful exchange of ideas

Research evidence presented in chapter 5 indicated that students' sense making involves a process of shared negotiation of meaning. In discussion-intensive learning environments, a high proportion of content development occurs through collective argumentation and group discussion. Sharing strategies and being involved in collaborative activities does not, however, necessarily result in students engaging in practices that support the development of mathematical understanding. Shaping productive mathematical dialogue involves establishing classroom sociomathematical norms that are specific to students' mathematical activities (Cobb et al., 1993; Yackel & Cobb, 1996). In CASE 6, classroom exchanges centred on fraction tasks illustrate how sociomathematical norms govern classroom discussions and potential opportunities for learning.

CASE 6: Sharing biscuits

(from Kazemi and Stipek, 2001)

Mathematics teaching for diverse learners:

- creates a space for the individual and the collective;
- provides opportunities for children to resolve cognitive conflict;
- involves the respectful exchange of ideas;
- involves explicit instructional discourse;
- provides opportunities for cognitive engagement and press for understanding.

Kazemi and Stipek (2001) observed the interactions within four teachers' lessons. They found that, despite the outward appearance in all classrooms of a focus on understanding, opportunities for students to participate in mathematical enquiry varied across the classrooms. Their study highlights those sociomathematical norms that promote students' engagement in conceptual mathematical thinking and conversations.

Targeted learning outcomes

The desired outcome of these tasks was for students to construct mathematical understanding about addition of fractions and to become skilled at communicating in mathematical language as they described and defended their differing mathematical interpretations and solutions.

Learning context

The study involved four teachers of grade 4 and 5, primarily low-decile classes, all teaching the same lesson on addition of fractions. The lesson involved the partitioning of brownie biscuits. The lesson plan provided one sample problem based on equivalence. Tools suggested for student use included sheets of paper with 16 pre-drawn squares.

On the surface, students appeared to be focused on understanding mathematics. However, a closer analysis of the interactions revealed differences in the way in which students were engaged in mathematical practices. In two of the four classrooms, the researchers categorised the interactions as consistently creating a high press for conceptual thinking. The other two classrooms were characterised as demanding a lower press for conceptual thinking.

The students were required to persuade each other by clear explanation and reasonable argument of their answer. In establishing rules of group participation, the teacher tried to ensure that the arguments for the errors were clearly voiced and justified.

In this episode, the teacher could have stepped in and pointed out why $6/8$ and $11/8$ are not equal. Instead, her response was to encourage her students to explore the error by explaining why $6/8$ and $11/8$ can't be equal.

Kazemi and Stipek also noted that Ms. Carter frequently used her observations of inadequate solutions during group work to plan whole-class instruction. In contrast, although incomplete or inadequate solutions were accepted as a normal part of learning in the other two classrooms, they were more likely to be ignored by the teacher or passed over until an adequate solution was offered or corrected by the teachers themselves.

Collaborative argumentation

In classes where there was a high press for understanding, collaborative work was accompanied by an expectation that each student was accountable for thinking through the mathematics involved in a problem. There was an expectation that consensus should be reached through mathematical argumentation. Teachers in both classrooms provided guidelines for small-group participation and reinforced these expectations. When Ms. Martin began the work on the fair-share problems, she made the following statement regarding individual accountability:

| Teacher: | Everyone in your group, whether it's just the two of you, or the three of you, everyone in your group needs to understand the process that you all were supposed to go through together. Because when you make a presentation, you don't know whether or not you are going to be asked a question. So you don't know if you're going to be asked by me or by your classmates. So you need to make sure that each person understands each part of the process you went through. |

When working in groups, these expectations appeared to have been understood in terms of distribution of labour and contribution. The following episode is typical of an interactive discussion:

Keisha:	See, this is how I explained it. [Reads.] "What we did is we took three brownies and cut them into half because three plus three equals six. And there are six crows."
Mark:	This is what I put so far. [Reads.] "We knew that there were three more brownies, and we divided each one in half. One brownie had two halves, and another brownie had two halves, and another one had two halves."
Keisha:	Just write, "And all three had two halves." And all ...
Mark:	[Starts writing.] "And all three ..." I don't have to write.
Keisha:	Okay, and each crow got $1/2$ and ... just write: "3 + 3 is 6" so each crow got $1/2$ of the brownie.
Mark:	Yeah, but these are halves. [Counts halves in each brownie.] 2, 2.
Keisha:	Yeah, I know. 1, 2, 3. [Counts brownies.] Cut them in half, 1, 2, 3, 4, 5, 6. [Counts halves.] There were six crows. Each crow got $1/2$.
Mark:	[Writes "Each crow got one-half."]

Having written about one of the steps in the problem, Mark and Keisha proceeded to evaluate and expand their written explanations. We see in the above interaction how Mark indicated that he did not appear to understand how "3 + 3 = 6" applied. Keisha's explanation that she was referring not to the halves themselves, but that the number of halves corresponded to the number of crows appears to have facilitated a mutual understanding.

In contrast to the high-press classrooms, the teachers in the low-press classrooms only gave general instructions (such as "work with a partner" or "remember to work together") to support collaboration. The researchers noted frequent occurrences of unequal distribution of work within groups. Students who were unclear about what to do often withdrew and allowed another student to take over. In the following example, Ellen was excluded from participation. Without an opportunity to think about the problem, her role was one of 'listen and agree':

In this exchange, the teacher asked her students to reflect on what was unique about a particular group's solution strategy. Students' responses included both organisational ("they divided it into steps") and mathematical ("they divided it into sixths") aspects. The researchers noted that the focus on mathematical differences among shared strategies supported students' formation of mathematical connections between various solution paths.

In contrast, in the 'low press for understanding' classrooms, discussions focused on non-mathematical aspects of shared strategies. The sharing of strategies "looked like a string of presentations, each one followed by applause and praise" (p. 72). Links, if they were made, consisted of non-mathematical aspects. For example, in Ms. Andrew's class, a pair of students reported that a solution strategy involved cutting the brownies and distributing the pieces to each individual. Another student reported drawing lines from the fractional parts of the brownies to the individuals who received them. Although both solutions used partitioning strategies, they were accepted as mathematically different, based on the way they handed out the pieces.

Building on errors

In classrooms where student participation and contribution is valued, student errors provide entry points for further mathematical discussion. Errors, or inadequate or partial solutions, provide opportunities to reconceptualise a problem, explore contradictions in a solution approach, or try out alternative strategies.

In the following episode, Ms. Carter organised a discussion based on conflicting student responses to the problem. By encouraging the whole class to think about a range of possible solutions, she created an opportunity for all students to engage in mathematical analysis. Following on from an earlier class presentation in which Sarah and Jasmine offer a diagrammatic solution to the problem of dividing nine brownies equally among eight people (fig. 7.8), Ms. Carter is involved in the following classroom exchange:

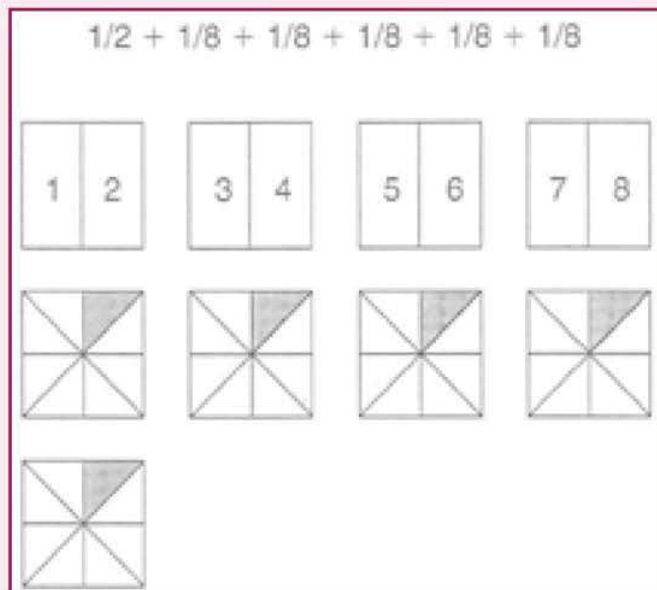| Ms. C: | Do you want to write that down at the top [of the OHP] so I can see what you did? [Jasmine writes $1/2 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8$] |
Ms. C:	Okay, so that's what you did. So how much was that in all?
J:	It equals $1\frac{1}{8}$ or $6/8$
Ms. C:	So she says it can equal 6 [sic] and $6/8$?

Fig. 7.8. Sharing 9 brownies among 8 people

Ms. Andrew did not ask students to justify why they chose a particular partitioning strategy. A commonly observed practice in Ms. Andrew's class was for students to indicate with a show of hands or by calling out 'yes' or 'no', their responses to questions such as "How many people agree?" "Does this make sense?" or "Do you think that was a good answer?" These general responses, while on the surface indicating participation, revealed limited information about students' thinking or their understanding of the mathematical concepts involved.

Understanding relationships

Sharing solution strategies enables students to reflect on relationships within mathematics. The following vignette illustrates how one of the teachers in Kazemi and Stipek's study supported her students' examination of the mathematical similarities and differences among multiple strategies.

After Michelle and Sally had described their strategy and solution for a fair-sharing problem (see figure 7.7), the teacher turned to the class:

|	
Ms. M:	Does anyone have any question about how they proceeded through the problem? ... What did they use or do that was different than what you might have done?
Jeff:	They used steps.
Ms. M:	Right, they divided it into steps. But there were some steps that I haven't seen anyone else use in the classroom yet.
Carl:	They added how many brownies there were altogether.
Ms. M:	Okay, so they used ...
Jan:	They divided into six, and there was one left over, and then they figured how they were going to divide that equally so that every crow gets a fair share.
Ms. M:	Exactly, and that was very observant of you to see that. As I walked around yesterday, this is the only pair that used a division algorithm to determine that there was a whole brownie and a piece left over. So they did it in two different ways.
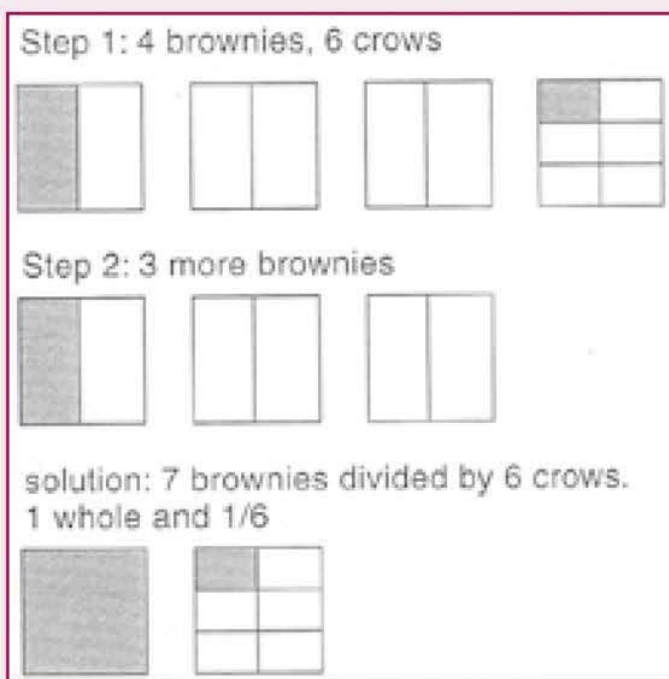
Fig. 7.7. Sharing 4 brownies among 6 crows and then 3 more brownies among 6 crows

Mathematical argumentation

This first vignette illustrates how students provided explanations that went beyond descriptions or summaries of the steps they used to solve the problem: they linked their problem-solving strategies to mathematical processes.

Luis: There were six crows, and we made, like, a colour dot on them … There were four brownies, and we divided three of them into halves and the last one into sixths. One of the crows got $\frac{1}{2}$ and $\frac{1}{6}$.

Chris: Each crow got $\frac{1}{2}$ and $\frac{1}{6}$. In our second step, we had three brownies and we divided them in half. So each crow got $\frac{1}{2}$. $\frac{1}{2}$ plus $\frac{1}{6}$ equals $\frac{4}{6}$. So we have $\frac{1}{2}$ and $\frac{1}{6}$ and right here is $\frac{4}{6}$. [He points to two squares: one divided into half and then $\frac{3}{6}$ and the other into sixths. $\frac{4}{6}$ had been shaded in each brownie; see figure 7.6.]

Luis: Just to prove that it's the same. Then $\frac{4}{6}$ is what they got here, plus $\frac{1}{2}$. And $\frac{1}{2}$ is equal to $\frac{3}{6}$. $\frac{4}{6}$ plus $\frac{3}{6}$ is equal to one whole and $\frac{1}{6}$.
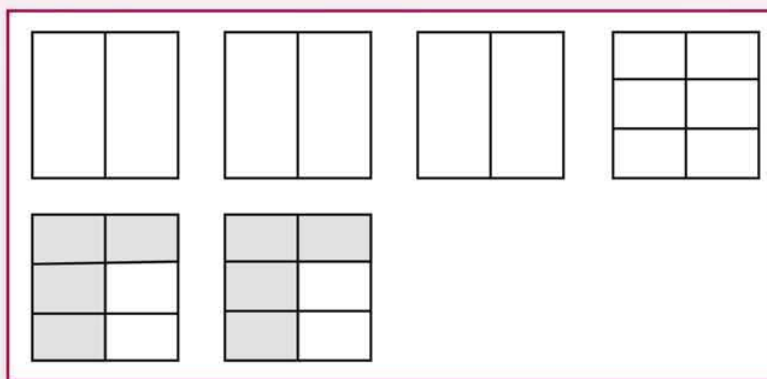

Fig. 7.6. Sharing 4 brownies among 6 crows to show that $\frac{1}{2}$ and $\frac{1}{6}$ equal $\frac{4}{6}$

It is clear from this episode that the students understood the need to demonstrate their mathematical argument for equivalence graphically as well as verbally. Kazemi and Stipek noted that the teacher invited everyone, not just the students at the board, to think about how the students had solved the problem. In the resulting discussion, the teacher required that students focus on the mathematical concept of equivalence and its relation to the process of adding fractional parts; it was not enough that students commented on the clarity of the drawings or how they were shaded.

To illustrate the effect of contrasting expectations, Kazemi and Stipek provide an episode from another classroom in which students engaged in the same social practices of describing their thinking but merely summarised the steps they took to solve a problem. The episode follows on from Raymond's description of his solution for dividing 12 brownies among eight people. The teacher, Ms. Andrew, had drawn 12 squares on the chalkboard. Raymond divided four of the brownies in half.

Ms. A: Okay, now would you like to explain to us what …

R: Each one gets one, and I give them a half.

Ms. A: So each person got how much?

R: One and $\frac{1}{2}$.

Ms. A: $\frac{1}{2}$?

R: No, one and $\frac{1}{2}$.

Ms. A: So you're saying that each one gets one and $\frac{1}{2}$. Does that make sense? [Chorus of "yeahs" from students; the teacher moves on to another problem.]

Lisa:	We need five brownies. So see, 1, 2, 3, 4. So we cut these into half. So 1 ... 8. They get a half each. And then there's one more cookie and eight people, so we just cut into eighths, and it'll be even for everybody.
Ellen:	Wow, you did that fast. I didn't even do anything.
Lisa:	I knew there's 5, and I knew 4; 2 times 4 is 8.
Ellen:	Oh, I get it.

Learner outcomes

Students in classrooms with established social and sociomathematical norms for mathematical thinking were more likely to be observed engaging in mathematical discussion that was conceptual rather than procedural in nature.

Quality pedagogy

The difference between the high- and low-press exchanges illustrates that, to support conceptual thinking, teaching needs to go beyond superficial practices based on the discussion and sharing of strategies. Kazemi and Stipek argue that pedagogical practices need to be based on the following sociomathematical norms that work together to establish expectations about what constitutes mathematical thinking:

- An explanation consists of a mathematical argument, not simply a procedural description or summary.

- Mathematical thinking involves understanding relations among multiple solutions.

- Errors provide opportunities to reconceptualise a problem, explore contradiction in solutions, or pursue alternative strategies.

- Collaborative work involves individual accountability and reaching consensus though mathematical argumentation.

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